1. Vector tangente como derivada direccional

Calculamos las derivadas parciales:

$$\frac{\partial f}{\partial x} = 2xy + y\cos(xy), \qquad \frac{\partial f}{\partial y} = x^2 + x\cos(xy).$$

En el punto $(1, \pi)$:

$$\frac{\partial f}{\partial x} = 2\pi + \pi\cos(\pi) = 2\pi - \pi = \pi, \quad \frac{\partial f}{\partial y} = 1 + \cos(\pi) = 1 - 1 = 0.$$

Entonces:

$$\mathbf{v}(f) = 2 \cdot \pi - 1 \cdot 0 = 2\pi.$$

2. Vector vs. campo vectorial

(a) $\mathbf{V}(2,3) = 2\frac{\partial}{\partial x} + 3\frac{\partial}{\partial y}$.

(b) Calculamos:

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 2y \Rightarrow \mathbf{V}(f) = x \cdot 2x + y \cdot 2y = 2x^2 + 2y^2.$$

En $(2,3)$: $\mathbf{V}(f) = 2 \cdot 4 + 2 \cdot 9 = 8 + 18 = 26$.

3. Campo vectorial actuando sobre una función

$$\frac{\partial f}{\partial x} = ye^{xy}, \quad \frac{\partial f}{\partial y} = xe^{xy} \Rightarrow \mathbf{V}(f) = y^2e^{xy} + x^2e^{xy} = (x^2 + y^2)e^{xy}.$$

En $(1,2)$: $\mathbf{V}(f) = 5e^2$.

4. Conmutador de campos vectoriales

$$\mathbf{Y}(f) = x\frac{\partial f}{\partial y}, \quad \Rightarrow \mathbf{X}(\mathbf{Y}(f)) = \frac{\partial}{\partial x}(x\frac{\partial f}{\partial y}) = \frac{\partial f}{\partial y} + x\frac{\partial^2 f}{\partial x \partial y}.$$

$$\mathbf{X}(f) = \frac{\partial f}{\partial x}, \quad \Rightarrow \mathbf{Y}(\mathbf{X}(f)) = x\frac{\partial^2 f}{\partial y \partial x}.$$

Entonces:

$$[\mathbf{X}, \mathbf{Y}](f) = \mathbf{X}(\mathbf{Y}(f)) - \mathbf{Y}(\mathbf{X}(f)) = \frac{\partial f}{\partial y}.$$

Así que $[\mathbf{X}, \mathbf{Y}] = \frac{\partial}{\partial y}$.

5. Covectores (vectores del espacio cotangente)

$$\frac{\partial f}{\partial x} = 2xy, \quad \frac{\partial f}{\partial y} = x^2 + 3y^2 \Rightarrow df = 2xy\,dx + (x^2 + 3y^2)\,dy.$$

En $(1,1)$: $df = 2\,dx + 4\,dy$.

$$\text{Aplicación: } df(\mathbf{v}) = 2 \cdot 1 + 4 \cdot 2 = 2 + 8 = 10.$$

6. Evaluar un covector sobre distintos vectores

$$\omega(\mathbf{v}_1) = 3, \quad \omega(\mathbf{v}_2) = -1, \quad \omega(\mathbf{v}_3) = 3 - 1 = 2.$$